Fundamentals of Machine Learning

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Maximal Inequality for bounded losses

Proposition : Upperbounding the max of bounded r.v.

Let $X_1, \ldots, X_n$ be n centered random variables $\left(\mathbb{E}\left[X_i\right] = 0 \text{ for all } i\right)$, such that $X_i \in [a, b]$ almost surely for all $i \in \{1, \ldots, n\}$.

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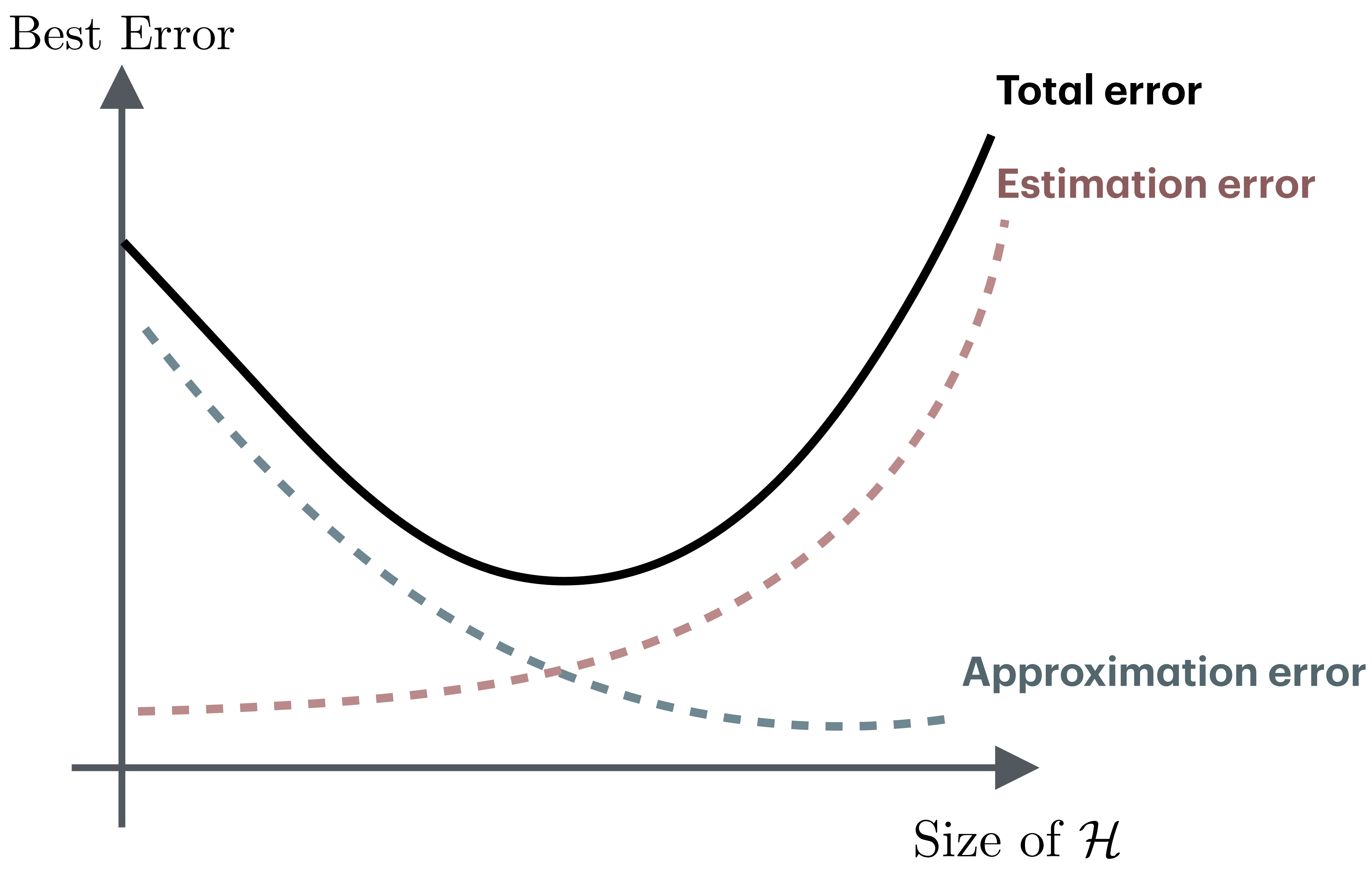
- When $|\mathcal{H}|$ is infinite, this bound is not exploitable.

1. Supervised Learning Setting

2. Estimation vs Approximation

3. Maximal inequalities

4. Rademacher Complexity



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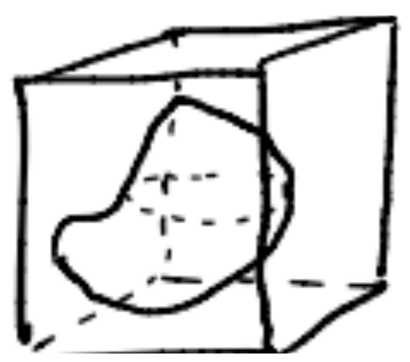

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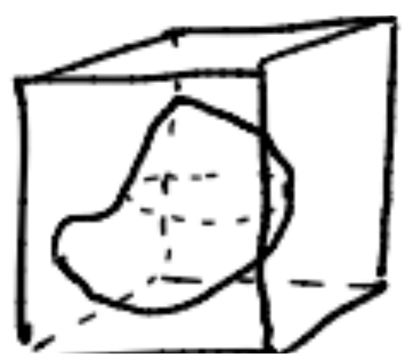

Proposition : Talagrand's lemma

Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be n G-lipchitz functions.

Let $T \subset R^n$ and define $(f_1, \ldots, f_n) \circ T$ as

$$(f_1, \ldots, f_n) \circ T = \{(f_1(t_1), \ldots, f_n(t_n)), t \in T\}.$$

Then,

$$\mathrm{Rad}((f, \ldots, f_n) \circ T) \leqslant G\,\mathrm{Rad}(T).$$

More properties

Proposition : Convex hull

Let $T \subset \mathbb{R}^n, v \in \mathbb{R}^n, c \in \mathbb{R}$, and define $cT + v = \{ct + v, t \in T\}$.

Then,

$$\text{Rad}(\text{conv}(T)) = \text{Rad}(T).$$

Proof : on the black board.

Proposition : Massart's Lemma - Complexity for finite Sets

Let $T \subset \mathbb{R}^n$ be finite.

Then,

$$\text{Rad}(T) \leq \max_{t \in T} \|t\|_2 \frac{\sqrt{2\log(|T|)}}{n}.$$

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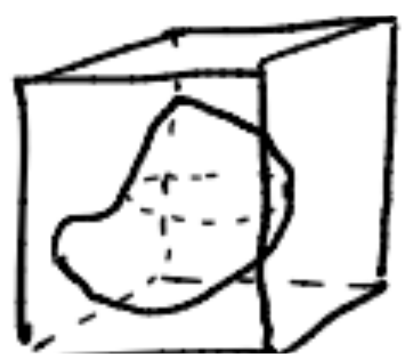

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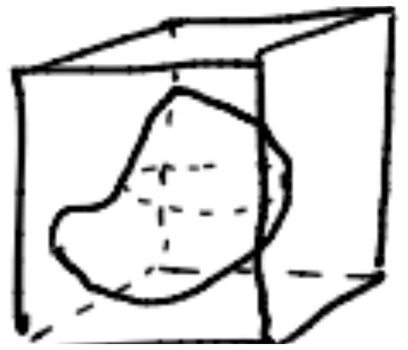

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Remark :

- This lemma will later be used to establish statistical bounds when the loss ℓ satisfies a Lispchitz condition.

Rademacher complexity of a hypothesis class

- Formal definition & Symmetrization Lemma

Definition : Rademacher complexity of a class of functions

Let $(\Omega_1, \ldots, \Omega_n)$ be n i.i.d. Rademacher variables.

We define the empirical Rademacher complexity of the hypothesis class $\mathcal{H}$, denoted $\widetilde{\mathrm{Rad}}(\mathcal{H})$, as

$$\widetilde{\mathrm{Rad}}(\mathcal{H}) = \mathbb{E}\left[\sup_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{N} \Omega_i \ell\left(f\left(X_i\right), Y_i\right) \mid \left(X_1, Y_1\right), \ldots, \left(X_n, Y_n\right)\right],$$

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Under the Empirical Risk Minimization Framework, we have :

$$\mathbb{E}\left[\sup_{f \in \mathcal{H}} \{r(f) - R(f)\}\right] \leqslant 2 \, \mathrm{Rad}(\mathcal{H}).$$

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Proposition : Complexity for Lipschitz Losses

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Remarks :

- Hence, assuming assuming a Lipschitz error function allows us to focus on the complexity spanned by the set of estimators rather than the composition $\ell(f(X), Y)$.